Final Review

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1. Limits.

1.1. Calculation

- $\lim\limits_{x \to x_0} x^a = x_0^a$ for any real number a and any point x_0 so that x_0^a is defined;
- $\lim\limits_{x \to x_0} e^x = e^{x_0}$ for any real number x_0;
- $\lim\limits_{x \to x_0} \ln x = \ln x_0$ for any real number $x_0 > 0$;
- $\lim\limits_{x \to x_0} \sin x = \sin x_0$ and $\lim\limits_{x \to x_0} \cos x = \cos x_0$ for any real number x_0;
- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$.

and the following rules of calculating limits:

- $\lim(cf + dg) = c \lim f + d \lim g$, where f, g are functions and c, d are real constants;
- $\lim(fg) = (\lim f)(\lim g)$ where f, g are functions so that their limits exist;
- $\lim \left(\frac{f}{g}\right) = \frac{\lim f}{\lim g}$ where f, g are functions so that their limits exist and the limit of g is non-zero;
- $\lim(f \circ g) = f(\lim g)$ if f, g are functions, f is continuous and the limit of g exists.

There's another way to compute limit: using the definition of derivatives. That is, if a limit is of the form

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

iv) $\displaystyle\lim_{x\to 1}\frac{\ln x-\ln 1}{x-1}$;

v) $\displaystyle\lim_{x\to -\infty}\frac{e^{2x}}{e^{x}+3e^{2x}}$ *and* $\displaystyle\lim_{x\to +\infty}\frac{e^{2x}}{e^{x}+3e^{2x}}$.

Solution: iv) Special limit from Sec. 5.4: $\boxed{e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}}$

$$\lim_{x\to 1}\frac{\ln x-\ln 1}{x-1}=\lim_{x\to 1}\ln x^{\frac{1}{x-1}}=\lim_{x\to 1}\ln\left(1+(x-1)\right)^{\frac{1}{x-1}}$$

$=\ln e=1$. ($\ln$ is continuous).

$$\frac{e^{2x}}{e^{x}+3e^{2x}}=\frac{e^{2x}\cdot e^{-2x}}{(e^{x}+3e^{2x})e^{-2x}}=\frac{1}{e^{-x}+3}.$$

v) $\lim_{x\to -\infty}e^{x}=0$, $\lim_{x\to +\infty}e^{x}=+\infty$.

$$\lim_{x\to -\infty}\frac{e^{2x}}{e^{x}+3e^{2x}}=\lim_{x\to -\infty}\frac{e^{x}}{1+3e^{x}}=\frac{\lim_{x\to -\infty}e^{x}}{\lim_{x\to -\infty}(1+3e^{x})}=\frac{0}{1+3\cdot 0}=0.$$

$$\lim_{x\to +\infty}\frac{e^{2x}}{e^{x}+3e^{2x}}=\lim_{x\to +\infty}\frac{1}{e^{-x}+3}=\frac{1}{3+0}=\frac{1}{3}.\ \square$$

1.2. Continuity & Differentiability

1.3 Definition. A function f is **continuous** at a real number a if $\lim_{x\to a} f(x) = f(a)$.

1.4 Definition. A function f is **differentiable** at a point a if there is a finite real number L so that $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = L$.

$f(x) = \frac{\text{- - - -}}{x^2}$ in $(-\infty, 0)$.

1.6 Problem. *Consider the function f defined by*

f is continuous in $(-\infty, 0)$ and in $(0, +\infty)$.

$$f(x) = \begin{cases} \frac{\sin 5x^2}{x} + 8, & \text{if } x < 0. \\ (a-b)x + 2a, & \text{if } x \geq 0 \end{cases}$$

1. *Determine the value of the constant a for which f is continuous at $x = 0$. You must carefully justify your answer.*

2. *Determine the values of the constants a and b for which f is differentiable at $x = 0$. You must carefully justify your answer.*

Solution. 1. $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\left(\frac{\sin 5x^2}{x} + 8\right) = 8 + \lim_{x\to 0^-}\left(\frac{\sin 5x^2}{5x^2}\cdot 5x\right) = 8 + 1\cdot 0 = 8.$

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left((a-b)x + 2a\right) = 2a$

f is continuous at $x=0$ if $\underbrace{\lim_{x\to 0^-} f(x)}_{8} = \underbrace{\lim_{x\to 0^+} f(x)}_{2a}$ so $2a = 8$, $\boxed{a=4}$.

$h < 0$.

$\boxed{\lim_{h\to 0^-} \frac{8-2a}{h}}$ is finite only if $\frac{8=2a}{}$ here.

we need $a=4$

2. $\lim_{h\to 0^-} \frac{f(h)-f(0)}{h-0} = \lim_{h\to 0^-} \frac{1}{h}\cdot\left(\overbrace{\boxed{\frac{\sin 5h^2}{h} + 8}}^{f(h)} \overbrace{-2a}^{f(0)}\right) = \lim_{h\to 0^-}\left(\frac{\sin 5h^2}{5h^2}\cdot 5 + \frac{8-2a}{h}\right) = 1\cdot 5 = 5$

$$\lim_{h\to 0^+} \frac{f(h)-f(0)}{h-0} = \lim_{h\to 0^+} \frac{1}{h}\left(\underbrace{\boxed{(a-b)h+2a}}_{f(h)} - \underbrace{\boxed{2a}}_{f(0)}\right) = a-b. = 4-b$$ f differentiable at $x=0$ if

$f(0)=2a$

$4-b=5$, that is, $\boxed{b=-1}$. $\square$

2. Derivatives.

2.1. Computations.

- $(x^a)' = ax^{a-1}$ for any real number a;
- $(\sin x)' = \cos x$ and $(\cos x)' = -\sin x$;
- $(a^x)' = a^x \ln a$ for any $a > 0$ but $a \neq 1$;

$$\left(G(x^2)\right)' = G'(x^2) \cdot (2x)$$
$$= \frac{x^2}{x^8+1} \cdot 2x$$

- $(\log_a x)' = \dfrac{1}{x \ln a}$ for any $a > 0$ but $a \neq 1$.

$$G'(x) = \frac{x}{x^4+1}$$

and using some derivation rules:

- $(cf + dg)' = cf' + dg'$ for any differentiable functions f, g and any constants c, d;
- $(fg)' = f'g + fg'$;
- $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$;
- $\left(f(g(x))\right)' = f'(g(x))g'(x)$.

$$\int_{\sqrt{x}}^{x^2} \frac{\theta}{\theta^4+1} d\theta = \int_a^{x^2} \frac{\theta}{\theta^4+1} d\theta + \boxed{\int_{\sqrt{x}}^{a} \frac{\theta}{\theta^4+1} d\theta}$$

$$= -\int_a^{\sqrt{x}} \frac{\theta}{\theta^4+1} d\theta.$$

3. $F(x) = \displaystyle\int_{\sqrt{x}}^{x^2} \frac{\theta}{\theta^4+1} d\theta;$

4. $y = \dfrac{\tan x}{x};$

5. $f(x) = (1+x)^{\frac{1}{x}}.$ $= e^{\left(\frac{1}{x}\ln(x+1)\right)}$

y

Solution: 3. $F(x) = \displaystyle\int_0^{x^2} \frac{\theta}{\theta^4+1} d\theta - \int_0^{\sqrt{x}} \frac{\theta}{\theta^4+1} d\theta.$

$= G(x^2) - G(\sqrt{x}),$ $\quad G(x) = \displaystyle\int_0^x \frac{\theta}{\theta^4+1} d\theta$

so $F'(x) = 2x G'(x^2) - \dfrac{1}{2\sqrt{x}} G'(\sqrt{x})$ ← chain rule.

$= 2x \cdot \dfrac{x^2}{x^8+1} - \dfrac{1}{2\sqrt{x}} \dfrac{\sqrt{x}}{x^2+1} = \dfrac{2x^3}{x^8+1} - \dfrac{1}{2(x^2+1)}$

$\left(I(g(x))\right)' = g'(x) \cdot I'(g(x))$ $\quad I(x) = \displaystyle\int_0^x h(t)dt.$

we know the derivative of $\displaystyle\int_0^{g(x)} h(t)\,dt$. is $g'(x) \cdot h(g(x)).$

5. $f(x) = e^{\frac{1}{x}\ln(x+1)}$, so $f'(x) = e^{\frac{1}{x}\ln(x+1)}\left(\frac{1}{x}\ln(x+1)\right)' = \left(\frac{-1}{x^2}\ln(x+1) + \frac{1}{x(x+1)}\right)e^{\frac{1}{x}\ln(x+1)}$

$\ln y = \frac{1}{x}\ln(1+x)$, then $\frac{1}{y}y' =$ $\qquad y' = (\quad)\cdot(1+x)^{\frac{1}{x}}$. $\square$

Office Hour : 1 - (possibly 7) pm Today,

10 - 12pm Tomorrow. (maybe earlier)

2-4pm SLH 200.

2.2. Implicit Differentiation.

2.3 Problem. *Consider the curve given by the equation*

$$\sin(xy) = \cos y + x.$$

Find the tangent line to this curve at the point $(1, \pi)$, *and use this to give an estimate of the* y*-value for a nearby point on the curve where* $x = 0.98$.

Solution. take derivative w.r.t. x:

$$\frac{d}{dx}\sin(xy) = \frac{d}{dx}(\cos y + x).$$

$$\cos(xy)\left(y + x\frac{dy}{dx}\right) = (-\sin y)\frac{dy}{dx} + 1.$$

$$(x\cos xy + \sin y)\frac{dy}{dx} = 1 - y\cos(xy).$$

$$\frac{dy}{dx} = \frac{1 - y\cos xy}{x\cos xy + \sin y}$$

when $x=1$ and $y=\pi$, $\displaystyle \frac{dy}{dx} = \frac{1-\pi\cos\pi}{1\cdot\cos\pi + \sin\pi} = \frac{1+\pi}{-1} = -(\pi+1)$

so the tangent line at $(1,\pi)$ is $y - \pi = -(\pi+1)(x-1)$.

linear approximation

$f(x+h) \approx f(x) + h\cdot f'(x)$

(for h very small)

In this case: Assume $f(x)$ is the function, want $f(0.98)$ $\quad 0.98 = x+h$.

write $x=1$, $h=-0.02$, then

$$f(x+h) \approx f(x) + h\cdot f'(x)$$

$$\begin{aligned} f(0.98) &= f(1) - 0.02\cdot f'(1) \\ &= \pi - (0.02)\cdot(-(\pi+1)) \\ &= \pi + 0.02(\pi+1) \\ &= 1.02\pi + 0.02. \quad \square \end{aligned}$$

2.3. Mean Value Theorems.

2.4 Theorem (Fermat). Let f be a function continuous on $[a, b]$ and differentiable in (a, b). If $a < c < b$ is an extreme point of f, then $f'(c) = 0$.

2.5 Theorem (Rolle). Let f be a function continuous on $[a, b]$ and differentiable in (a, b) so that $f(a) = f(b)$, then there is $a < c < b$ such that $f'(c) = 0$.

2.6 Theorem (Mean Value Theorem). Let f be a function continuous on $[a, b]$ and differentiable in (a, b), then there is a real number $a < c < b$ so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

There's a theorem not quite related to derivatives, but we always combine these results together to solve problems.

2.7 Theorem (Intermediate Value Theorem). If f is continuous in the interval $[a, b]$ and $f(a)f(b) < 0$, then there exists $a < c < b$ such that $f(c) = 0$.

And there's an existence theorem about absolute extrema:

2.8 Theorem. If f is a continuous function on $[a, b]$, then f must have an absolute maximum and an absolute minimum.

2.9 Problem. *Show that* $\sqrt{1+x} \le \sqrt{2} + \dfrac{x-1}{2\sqrt{2}}$ *for* $x \ge 1$.

Proof: $\sqrt{1+x} \le \sqrt{2} + \frac{x-1}{2\sqrt{2}}$ is the same as $\sqrt{1+x} - \sqrt{2} \le \frac{x-1}{2\sqrt{2}}$ if $x>1$,

divide both sides by $x-1$ and because $x-1>0$, (if $x=1$, then $\sqrt{2} \le \sqrt{2}$)

$$\frac{\sqrt{1+x}-\sqrt{2}}{x-1} \le \frac{1}{2\sqrt{2}}.$$

how we think of this.

what we actually write down.

if we assume $f(x)=\sqrt{1+x}$, then $f(1)=\sqrt{2}$, so we get by mean value theorem, a $\exists$

$1<c<x$ so that $\boxed{\dfrac{f(x)-f(1)}{x-1} = f'(c)} = \dfrac{1}{2\sqrt{1+c}} < \dfrac{1}{2\sqrt{2}}$.

$c>1$,

$c+1>2$

$\sqrt{c+1} > \sqrt{2}$.

$\frac{1}{\sqrt{c+1}} < \frac{1}{\sqrt{2}}$.

$f(x)=\sqrt{1+x}$

$f'(x) = \frac{1}{2\sqrt{1+x}}$.

$\frac{1}{2\sqrt{1+x}} <$

$\dfrac{f(x)-f(1)}{x-1} = \dfrac{\sqrt{1+x}-\sqrt{2}}{x-1}$.

$x>c>1$, so $\frac{1}{2\sqrt{1+c}} < \frac{1}{2\sqrt{2}}$

so we get $\dfrac{\sqrt{1+x}-\sqrt{2}}{x-1} < \dfrac{1}{2\sqrt{2}}$ when $x>1$, so $\sqrt{1+x}-\sqrt{2} < \frac{1}{2\sqrt{2}}(x-1)$.

so $\sqrt{1+x} < \sqrt{2} + \frac{x-1}{2\sqrt{2}}$ when $x>1$, & $\sqrt{1+x} \le \sqrt{2} + \frac{x-1}{2\sqrt{2}}$ when $x \ge 1$. $\square$

2.11 Problem. *Let* $f(x) = x^4 + x - 3$.

1. *Show that* $f(x)$ *has a root in the interval* $[-2, 0]$, *and a root in the interval* $[0, 2]$.

2. *Show that* $f(x)$ *does not have more than two roots.*

Proof: 1 $\boxed{f'(x) = 4x^3 + 1 \text{ has one root } -\sqrt[3]{\tfrac{1}{4}}.}$

$f(-2) = (-2)^4 - 2 - 3 = 16 - 2 - 3 = 11 > 0$

$f(0) = -3 < 0$

$f(2) = 2^4 + 2 - 3 = 15 > 0.$

$f'(x) = 0$ $\quad 4x^3 + 1 = 0$ $\quad x^3 = -\frac{1}{4}$

$4x^3 = -1,$ $\quad x = \sqrt[3]{-\frac{1}{4}}.$

by intermediate value thm,

f has a root in $[-2, 0]$

& a root in $[0, 2]$.

2. If f has at least 3 roots, then by Rolle's theorem, f' has at least two distinct roots, which is a contradiction b/c f' has only one root. □.

If we don't know the number of roots, then we need to determine the interval where f is increasing or decreasing

$x = -\sqrt[3]{\frac{1}{4}}$ a critical pt of f.

$(-\infty, -\sqrt[3]{\tfrac{1}{4}})$ $\downarrow$. at most 1

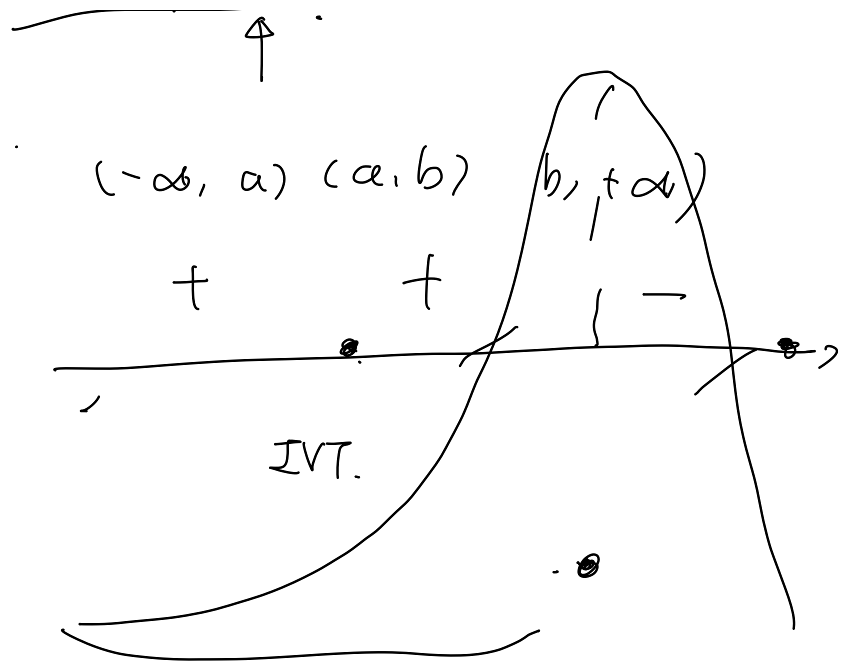
$(-\sqrt[3]{\tfrac{1}{4}}, +\infty)$ $\uparrow$ at most 1 at most 2 roots of f.

exactly 2 roots : use IVT.

$f\left(-\sqrt[3]{\frac{1}{4}}\right)$.

$f(-2)$

$f(2)$



Intermediate Value Thm

Mean Value Thm.

Find roots.

f' & f.

judge maximal possible # of roots / prove inequalities.

3. Curve Sketching.

3.3 Problem. *Consider the function* $f(x) = \dfrac{(x^4+1)^{\frac{1}{4}}}{1-x}$ *on the domain* $(-\infty, 1) \cup (1, +\infty)$.

1. *Investigate for the existence of horizontal and vertical asymptotes of the graph of* f. *Your answer must be supported by the careful calculation of relavant lmits.(**Hint:** $(x^4)^{\frac{1}{4}} = |x|$)*

2. $f'(x) = \dfrac{(x+1)(x^2-x+1)}{(1-x)^2(x^4+1)^{\frac{3}{4}}}$. *Note that* (x^2-x+1) *is always positive. Study the sign of* f', *then determine the intervals of increase, and of decrease of* f. *Indicate the values of local extrema, if any.*

3. $f''(x) = \dfrac{(x+1.64)}{1-x} M(x)$, *where* $M(x) > 0$. *Study the sign of* f'', *then determine the intervals where* f *is concave up, and where it is concave down. List all inflection points, if any.*

4. *Based on all the information gathered in the previous questions, sketch the graph of* f *as accurately as possible. Include all relevant facts as well as some remarkable points.(**Hint:** $2^{\frac{1}{4}} \approx 1.2$; $f(-1.64) \approx 0.65$)*

Solution :

$f(x) = \frac{(x^4+1)^{\frac{1}{4}}}{1-x}$, $f'(x) = \frac{(x+1)(x^2-x+1)}{(1-x)^2(x^4+1)^{\frac{3}{4}}}$, $f''(x) = \frac{(x+1.64)}{1-x} M(x)$, $M(x)>0$.

4. Applications.

4.3 Problem. *It's a hot day in L. A. and Carina has an ice cream cone. The ice cream is leaking into the cone at a rate of* $3/2cm^3$ *per second. Given that the cone is* $10cm$ *high, with a radius at the largest end of* $3cm$*, at the moment when the leaked ice cream fills half-way down the cone, what is the rate of change of the height of the liquid ice cream in the cone?(Hint: the formula for the volume of a right circular cone is* $V = \frac{1}{3}\pi r^2 h$ *where* r *is the radius of the cone, and* h *is the height.)*

Solution:

4.7 Problem. *A deposit of ore contains 100-mg of radium-226, which undergoes radioactive decay. After 500 years, 80.4% of the original mass of radium-226 remains.*

1. *Find the mass $m(t)$ of radium-226 that remains after t years.*
2. *What is the half-life of radium-226?*
3. *When will there be 20-mg of radium-226 remaining?*

exponential decay.

$m(t) = Ce^{\lambda t}$

$m(0) = C = 100$ mg.

$m(500) = Ce^{500\lambda}$

$= 80.4$ mg.

$80.4\% \cdot 100$ mg.

$\cancel{100}\, e^{500\lambda} = \cancel{100} \cdot 80.4\%$

$e^{500\lambda} = 80.4\%$.

$500\lambda = \ln(0.804)$

Solution: 1. $C = 100$, $e^{500\lambda} = 0.804$. $\lambda = \frac{1}{500}\ln(0.804)$

$m(t) = 100\, e^{\left(\frac{1}{500}\ln(0.804)\right)t}$.

2. $m(t) = 50$. $e^{\left(\frac{1}{500}\ln(0.804)\right)t} = \frac{1}{2}$.

$\frac{1}{500}\ln(0.804)t = \ln\frac{1}{2}$.

$t = -\ln 2 \cdot \frac{500}{\ln 0.804}$.

3. $m(t) = 20 \rightsquigarrow$ solve for t. $t = 500 \cdot \frac{\ln\frac{1}{5}}{\ln(0.804)}$

5. Quiz last time.

Problem 1. (8 points) Let

$$F(x) = -\int_{\frac{\pi}{4}}^{x^3} \ln(\sin t)\,\mathrm{d}\,t,\ 0 < x < \sqrt[3]{\pi}.$$

Show that F is invertible and find $(F^{-1})'(0)$. (The result will be a little bit complicated, believe in yourself!)

Proof & Solution :

Good Luck!